Introduction to Bayesian Inference

Osamu Maruyama

http://www.design.kyushu-u.ac.jp/~maruyama/summer-school-2019.pdf

Contents

Statistical inference

You lost a book yesterday.

A = “The book I lost yesterday should be on the desk of my house.”

A is 80%!

$P(A) = 0.8.$

Degree of belief.

Statistical inference

You guess today's dinner.

A = "The dinner today might be mapo dofu."

A is 10%!

$P($dinner today = mapo

Bayesian inference

Technique of statistical inference using a posterior probability distribution

$$P(\theta|D)$$

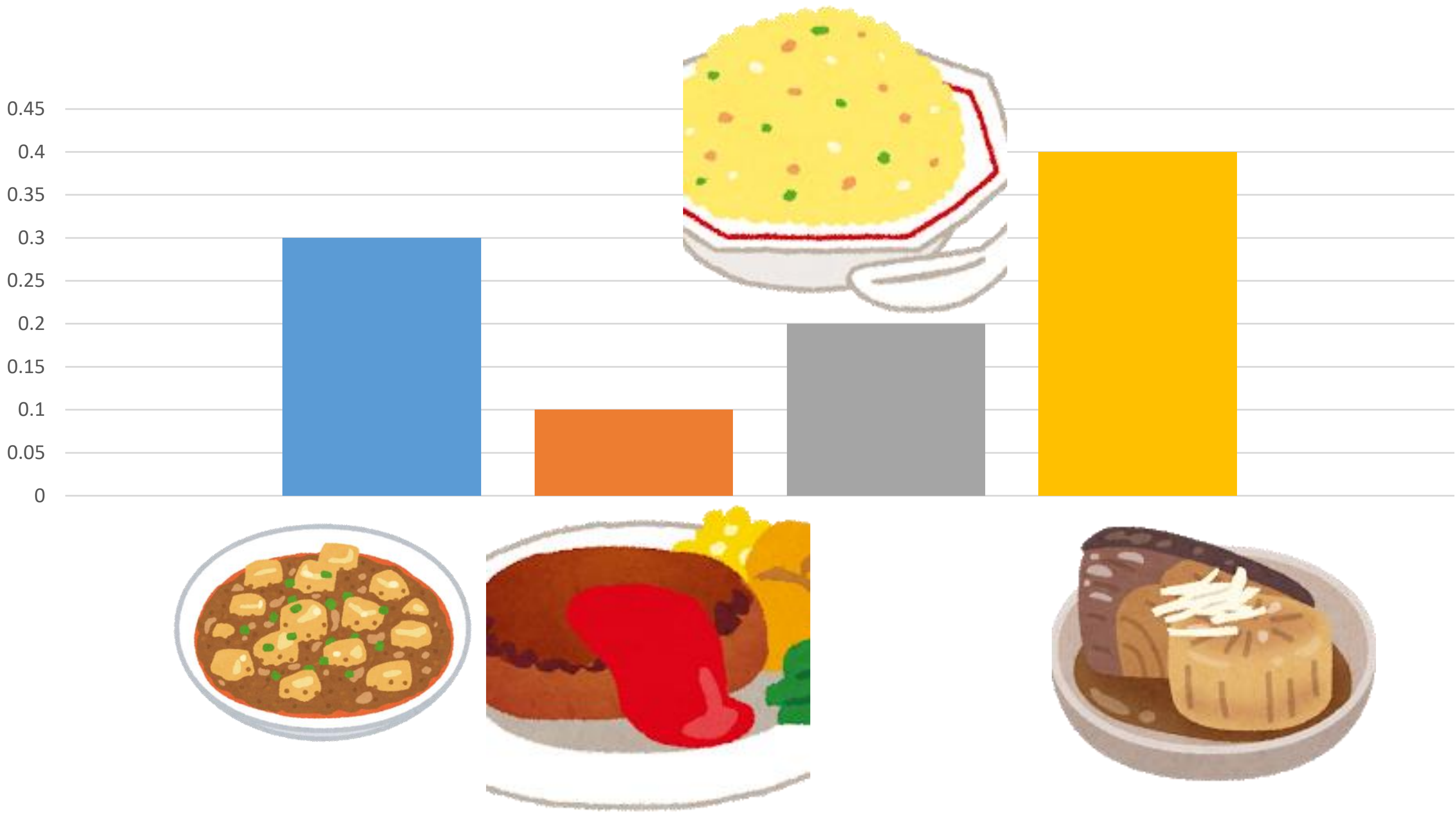
θ : random variable representing parameter, hypothesis

: **target objects to be estimated**

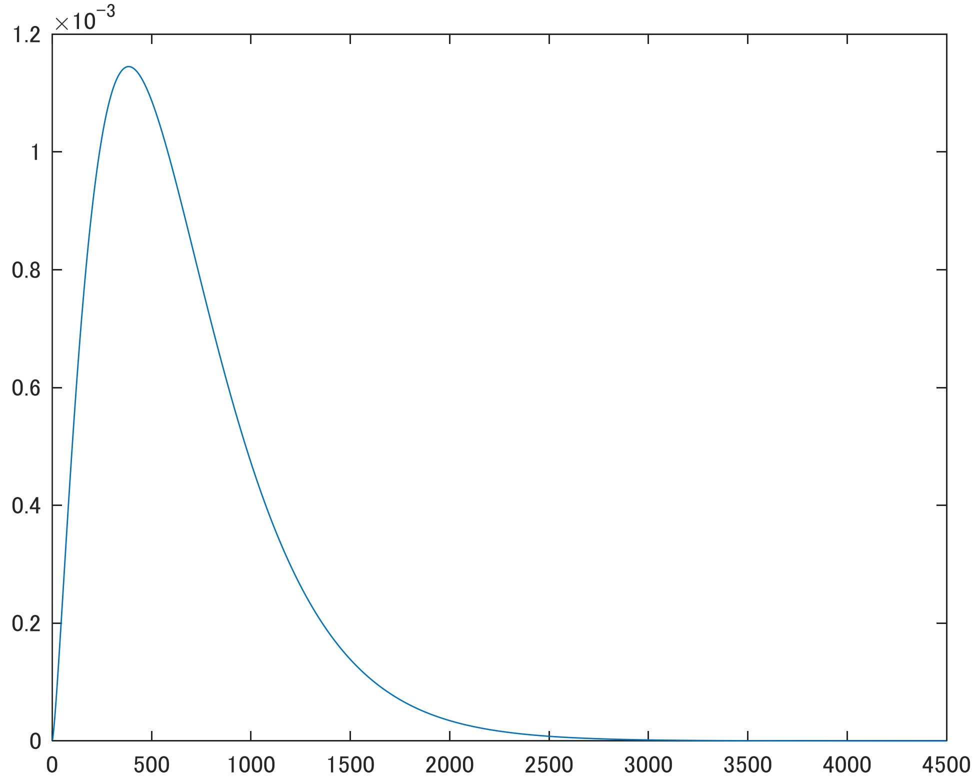
D : random variable representing observed data

: Observed data

Discrete random variable



Continuous random variable



Gamma distribution:

$$P(x|a,b) = \frac{1}{b^a \Gamma(a)} x^{a-1} e^{-\frac{x}{b}}$$

A probability is a function satisfying

$$0 \leq P(x) \leq 1$$

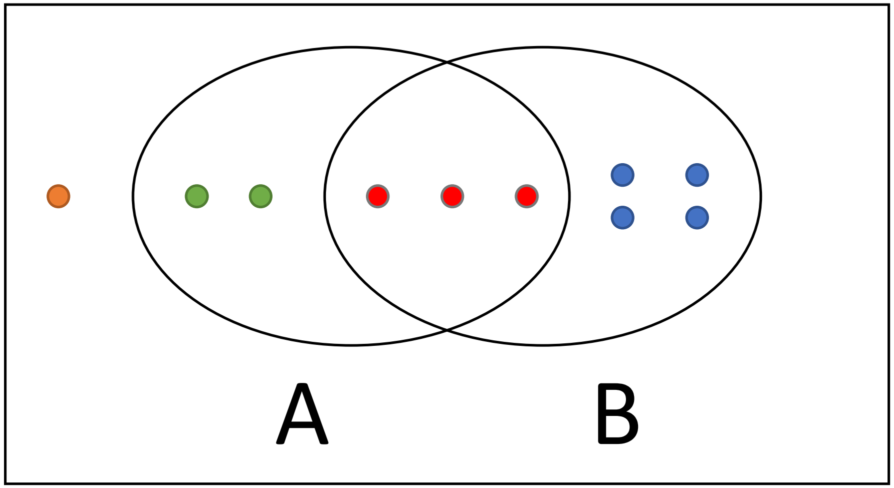
$$\sum_{x} P(x) = 1$$

Conditional probability

$$P(B|A) \qquad \left(= P_A(B)\right)$$

$$= \frac{P(A \cap B)}{P(A)}$$

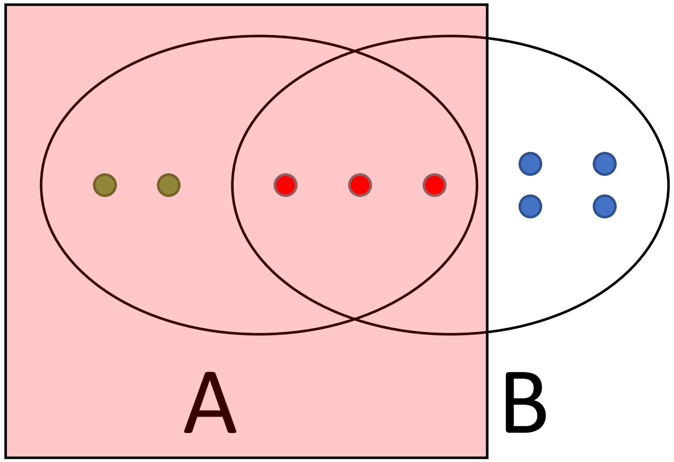
$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$



$$P(A) = \frac{5}{10}$$

$$P(B) = \frac{7}{10}$$

$$P(A \cap B) = \frac{3}{10}$$



$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{3}{10} \cdot \frac{10}{5} = \frac{3}{5}$$

← Compare the direct calculation of P(B|A)

Conditional probability

$$P(B|A)$$

$$\left(= P_A(B)\right)$$

$$= \frac{P(A \cap B)}{P(A)}$$

Multiplication theorem

$$P(A \cap B)$$

$$= P(B|A)P(A)$$

Bayes' theorem

Recall multiplication theorem:

$$P(A \cap B) = P(B|A)P(A)$$

$$\therefore P(A \cap B) = P(B \cap A) = P(A|B)P(B)$$

The two R.H.Ss are equal:

$$P(A|B)P(B) = P(B|A)P(A).$$

$$\therefore P(B|A) = \frac{P(A|B)P(B)}{P(A)}.$$

Key concepts of Bayes' theorem

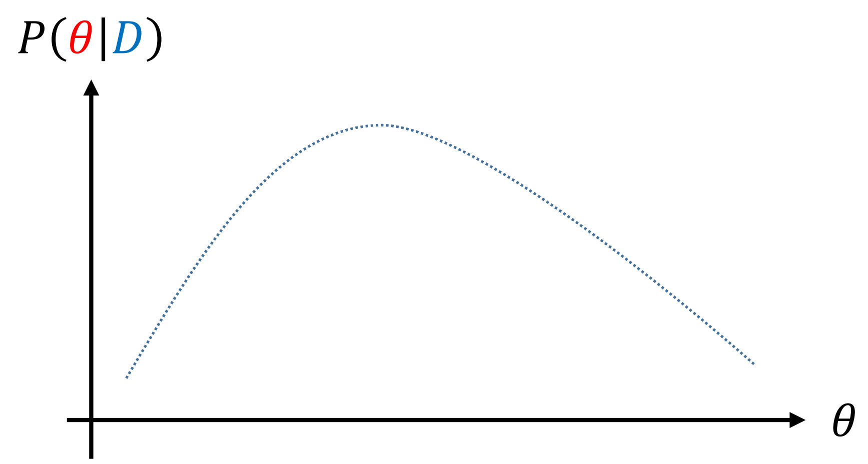
Bayes' theorem

$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)}$$

θ : parameter, hypothesis （representing causes）

: **target object to be estimated**

D : observed data



Bayes' theorem

$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)}$$

θ : parameter, hypothesis（cause）
D : observed data

- $P(\theta|D)$
  - posterior probability （事後確率）.
  - interpretation: probability of cause θ when event D happens.
- $P(D|\theta)$
  - likelihood function（尤度関数）
  - interpretation: likelihood of D under θ.
- $P(\theta)$
  - prior probability （事前確率）.
  - interpretation: a general degree of belief in θ.

Usefulness of Bayes' theorem

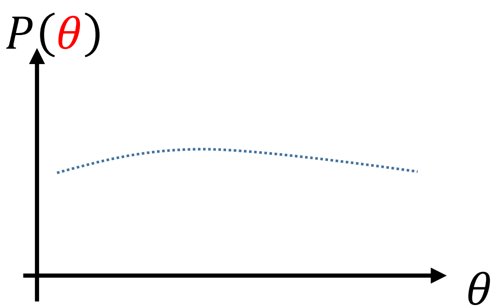
Current knowledge:

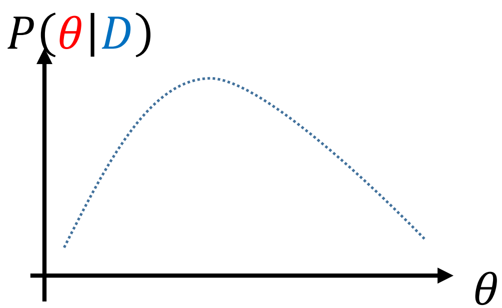
$$P(\theta)$$

D is observed

Updated knowledge:

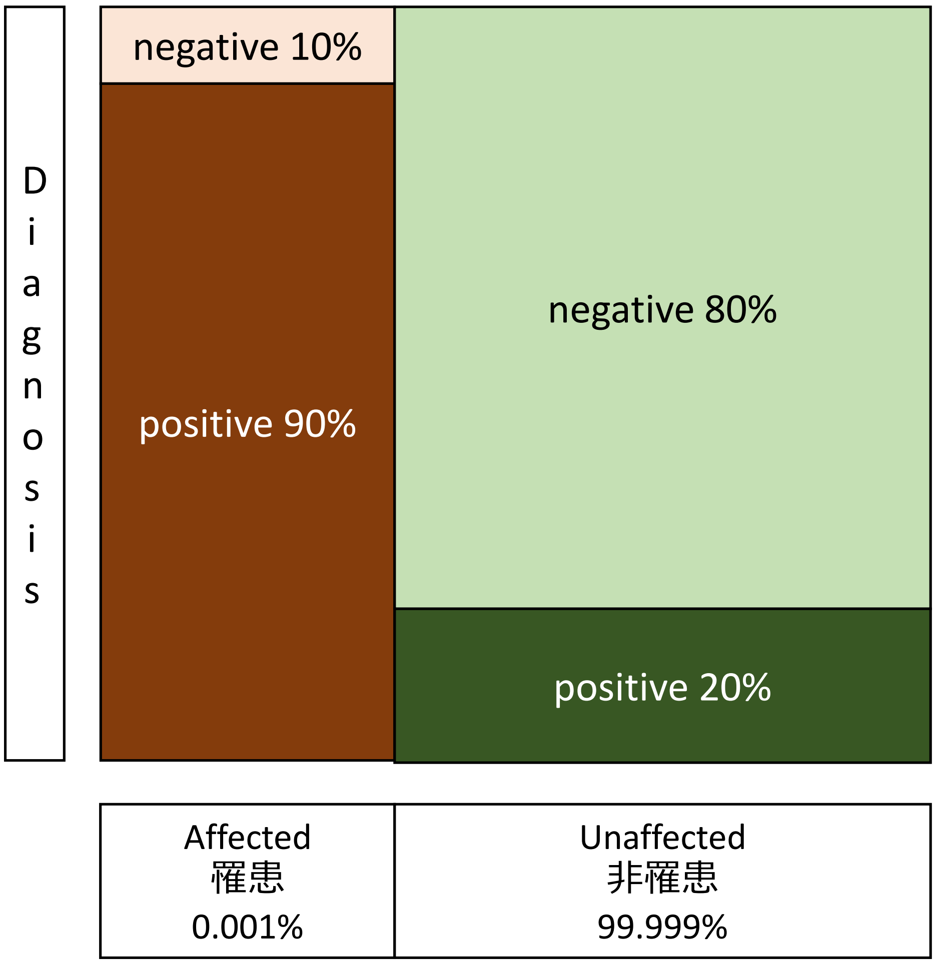
$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)}$$





Example: Diagnosis

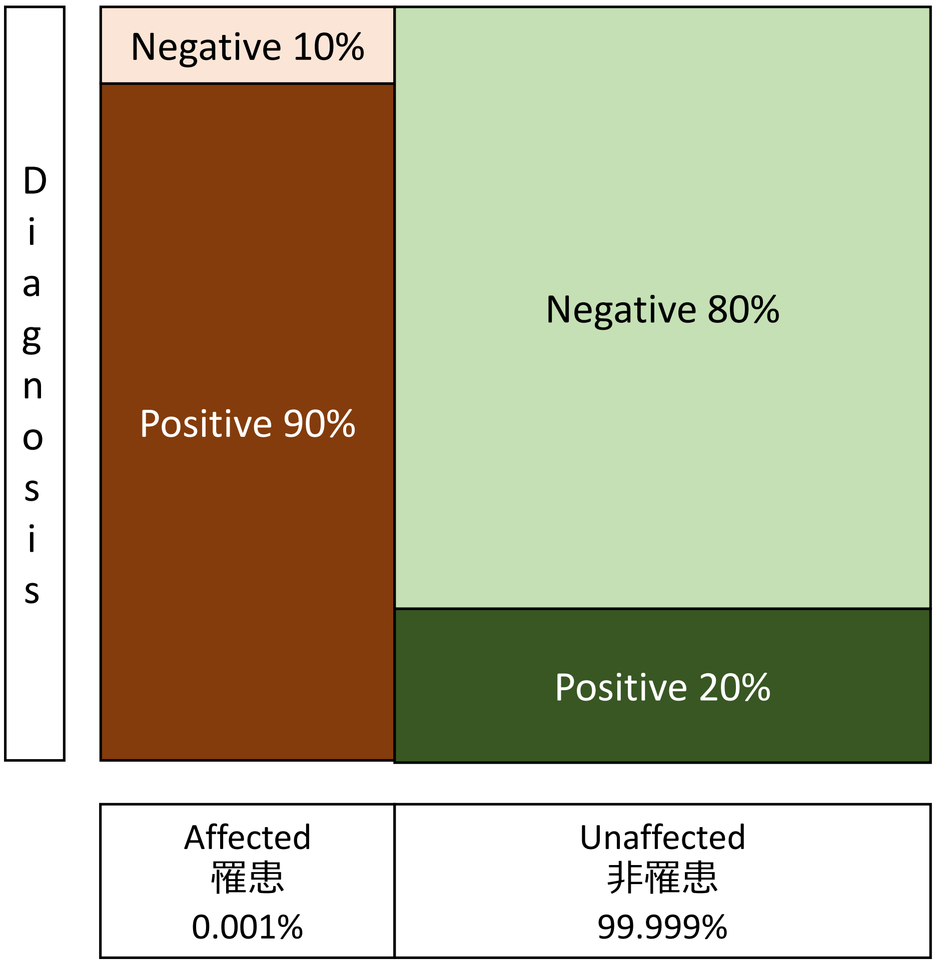
Example: Diagnosis



You are diagnosed positive.

Question 1:
Do you believe that you are affected?

Example: Diagnosis



You are diagnosed positive.

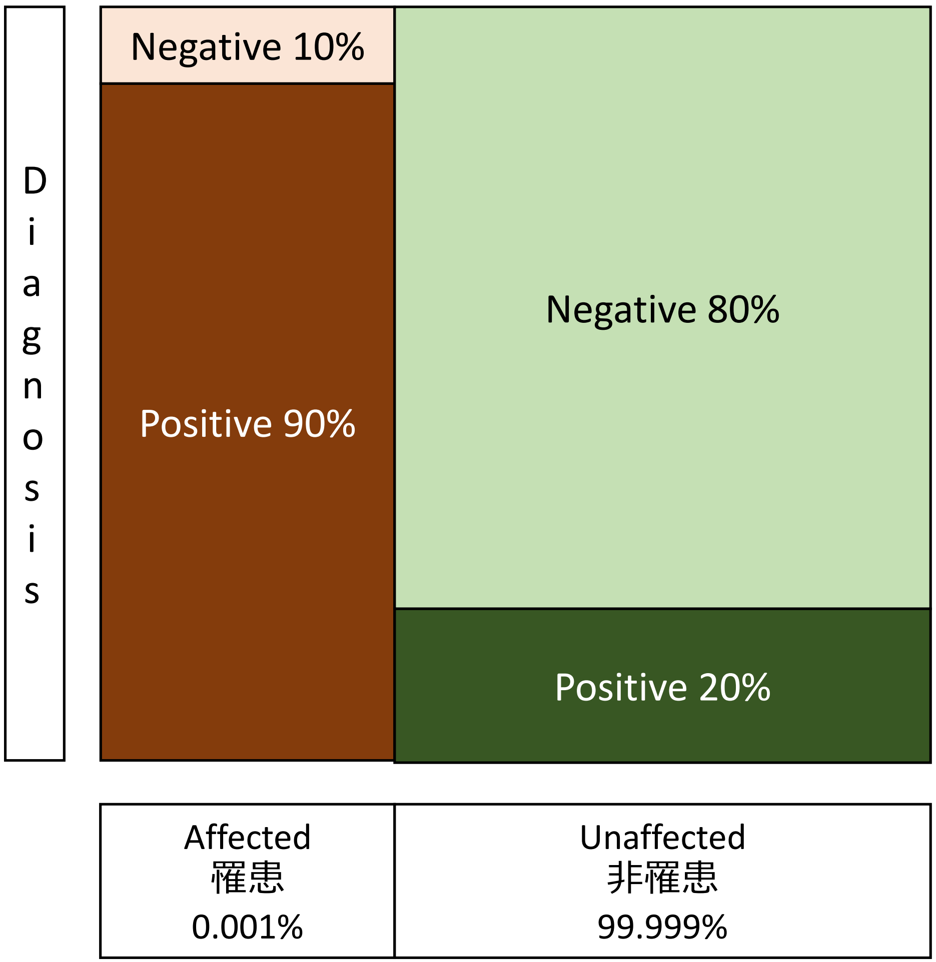
Question 2:
What's the probability that you are affected?

Luckily we have statistical data:

$A = \text{Affected}$
$U = \text{Unaffected}$
$P = \text{Positive}$
$N = \text{Negavive}$
$D = \{P, N\}$: random variable for diagnose
$R = \{A, U\}$: random variable for real state

$$P(R = A) = 0.00001$$
$$P(R = U) = 0.99999$$
$$P(D = P|R = A) = 0.9$$
$$P(D = N|R = A) = 0.1$$
$$P(D = P|R = U) = 0.2$$
$$P(D = N|R = U) = 0.8$$

Example: Diagnosis



A = Affected
U = Unaffected
P = Positive
N = Negavive
$D = \{P, N\}$: random variable for diagnose
$R = \{A, U\}$: random variable for real state

$$P(R = A) = 0.00001$$
$$P(R = U) = 0.99999$$
$$P(D = P|R = A) = 0.9$$
$$P(D = N|R = A) = 0.1$$
$$P(D = P|R = U) = 0.2$$
$$P(D = N|R = U) = 0.8$$

Which is larger?

$$P(R = A|D = P)$$
$$P(R = U|D = P)$$

$$P(R=A)=0.00001$$
$$P(R=U)=0.99999$$
$$P(D=P|R=A)=0.9$$
$$P(D=N|R=A)=0.1$$
$$P(D=P|R=U)=0.2$$
$$P(D=N|R=U)=0.8$$

Using Bayes' theorem

$$P(R=A|D=P)=\frac{P(D=P|R=A)P(R=A)}{P(D=P)}$$

$$=\frac{0.9\cdot 0.00001}{P(D=P)}=\frac{0.000009}{P(D=P)}$$

$$P(R=U|D=P)=\frac{P(D=P|R=U)P(R=U)}{P(D=P)}$$

$$=\frac{0.2\cdot 0.99999}{P(D=P)}\approx\frac{0.2}{P(D=P)}$$

Bayesian updating

- θ: probability of getting the head of a coin when it is flipped.
- Evaluate the value of θ as posterior probabilities!
- Likelihood function:
  - Bernoulli （ベルヌーイ） distribution with parameter θ
  - H: Head
  - T: Tail
  - $p(H|\theta) = \theta$
  - $p(T|\theta) = 1 - \theta$

Initial step of Bayesian updating

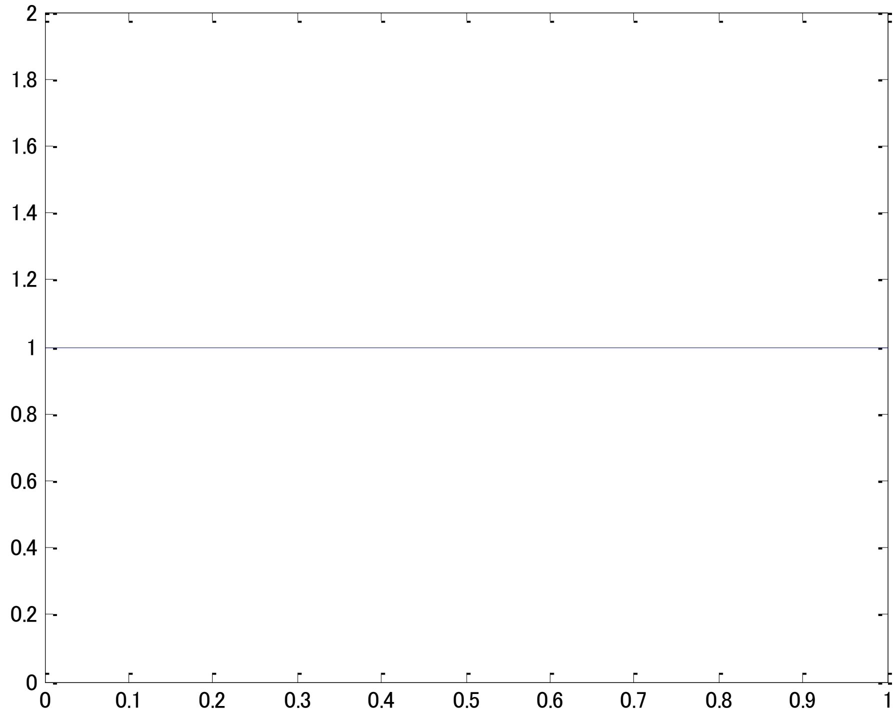
- No prior knowledge, assume a prior
- It is assumed

$$p(\theta) = constant.$$

$$\therefore p(\theta) = 1$$

p is a uniform distribution.

Current (initial) prior distribution $p(\theta) = 1$



This means we have no information on θ.

Suppose we have event
D_1: the head appeared

Find the posterior distribution $p(\theta|D_1)$

$$p(\theta|D_1) \propto p(H|\theta)p(\theta) = \theta \times 1 = \theta$$

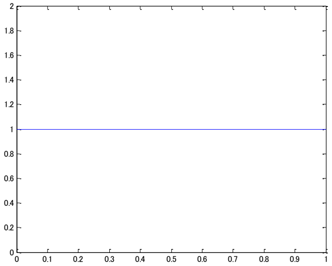
Normalization:

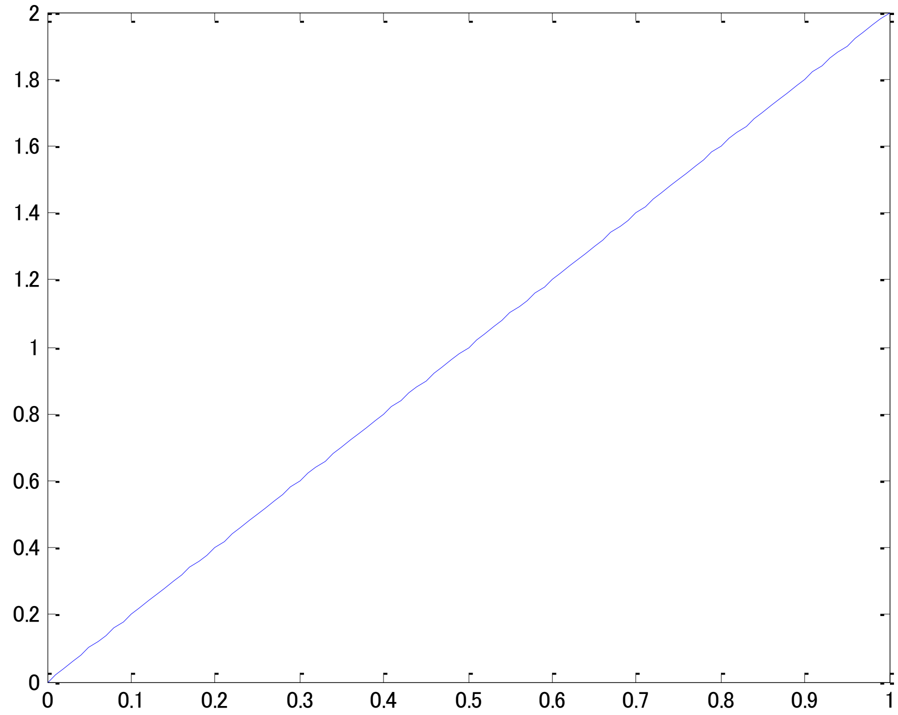
$$\int_0^1 p(H|\theta)p(\theta)\,d\theta = \int_0^1 \theta\,d\theta = \left[\frac{1}{2}\theta^2\right]_0^1 = \frac{1}{2}$$

We have

$$p(\theta|D_1) = 2\theta.$$

Posterior distribution $p(\theta|D_1) = 2\theta$





Reflecting D_1: the head appeared,
the higher θ is, the higher the probability is

Suppose we have event D_2: the head appeared

$$\begin{aligned} &p(\theta|D_2, D_1) \\ &\propto p(D_2|\theta)p(\theta|D_1) \\ &= p(H|\theta)p(\theta|D_1) \\ &= \theta \times 2\theta = 2\theta^2 \end{aligned}$$

The latest posterior distribution, $p(\theta|D_1)$ is used as the prior distribution in this step because it is the best knowledge of θ.

Normalization:

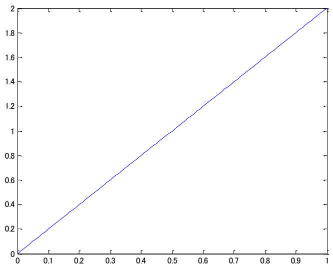
$$\int_0^1 2\theta^2\, d\theta = \left[\frac{2}{3}\theta^3\right]_0^1 = \frac{2}{3}$$

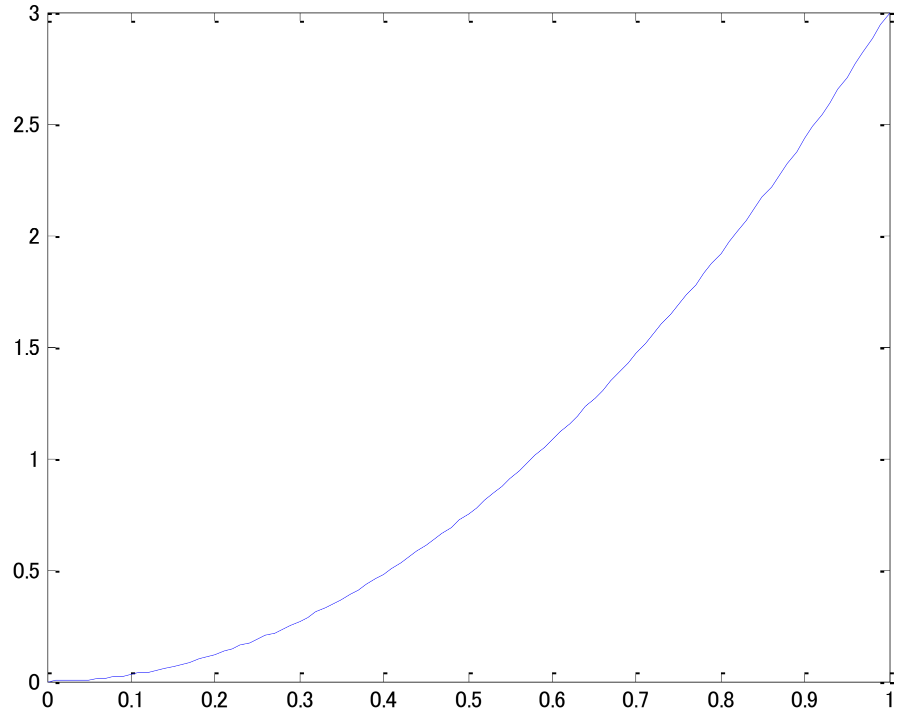
Thus, we have

$$p(\theta|D_2, D_1) = \frac{p(D_2|\theta)p(\theta|D_1)}{p(D_2|D_1)} = 3\theta^2$$

Posterior distribution

$$p(\theta|D_2, D_1) = 3\theta^2$$





This graph looks reasonable because we had $D_1 = D_2 = head$.

Suppose we have event D_3: the tail appeared

Find the posterior $p(\theta|D_3, D_2, D_1)$

$$\begin{gathered} p(\theta|D_3, D_2, D_1) \\ \propto p(D_3|\theta)p(\theta|D_2, D_1) \\ = p(T|\theta)p(\theta|D_2, D_1) \\ = (1-\theta) \times 3\theta^2 \end{gathered}$$

Normalization:

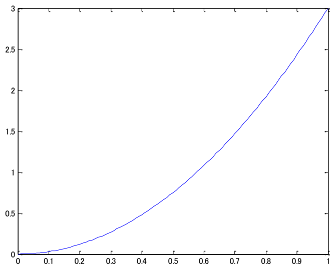
$$\int_0^1 (1-\theta) \times 3\theta^2 \, d\theta = \left[\theta^3 - \frac{3}{4}\theta^4\right]_0^1 = \frac{1}{4}$$

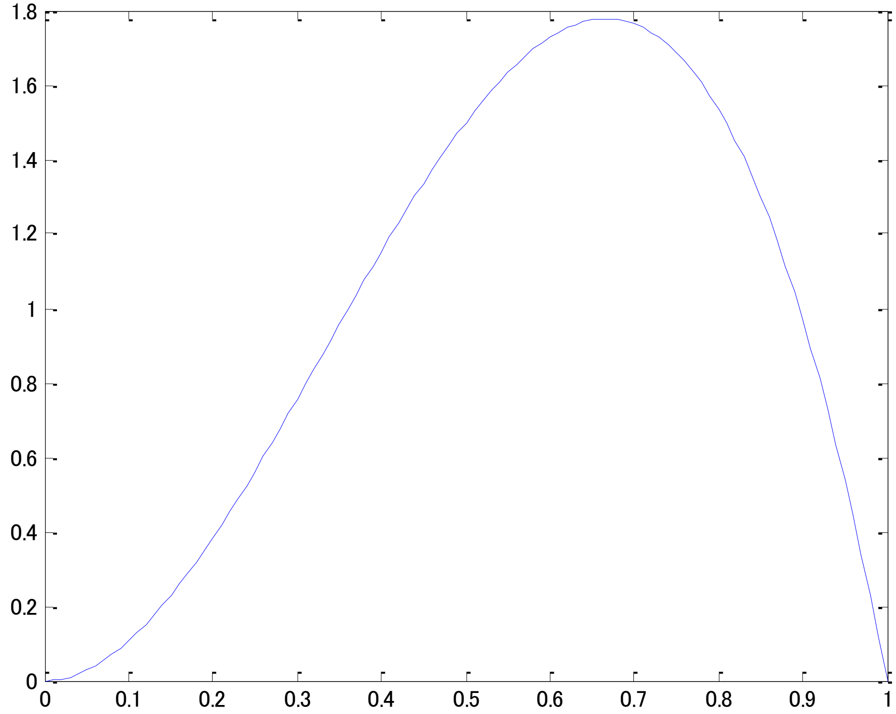
We have

$$p(\theta|D_3, D_2, D_1) = 12(1-\theta) \times \theta^2.$$

Posterior distribution

$$p(\theta|D_3, D_2, D_1) = 12(1-\theta) \times \theta^2$$





Let $f(\theta) = 12(1-\theta)\theta^2 = 12(\theta^2 - \theta^3)$.

$$f'(\theta) = 12(2\theta - 3\theta^2) = 12\theta(2 - 3\theta).$$

$$f'(\theta) = 0 \Leftrightarrow \theta = 0, \frac{2}{3}$$

The fact that the maximum value of the posterior is given at $\theta = \frac{2}{3}$ coincides with the current observations: H, H, T.

Suppose we have event
D4: the tail appeared

Find the posterior $p(\theta|D_4, D_3, D_2, D_1)$

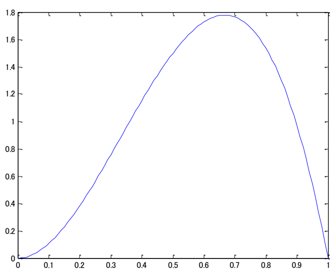
$$p(\theta|D_4, D_3, D_2, D_1)$$
$$\propto p(D_4|\theta)p(\theta|D_3, D_2, D_1)$$
$$= p(T|\theta)p(\theta|D_3, D_2, D_1)$$
$$= (1-\theta) \times 12(1-\theta) \times \theta^2$$

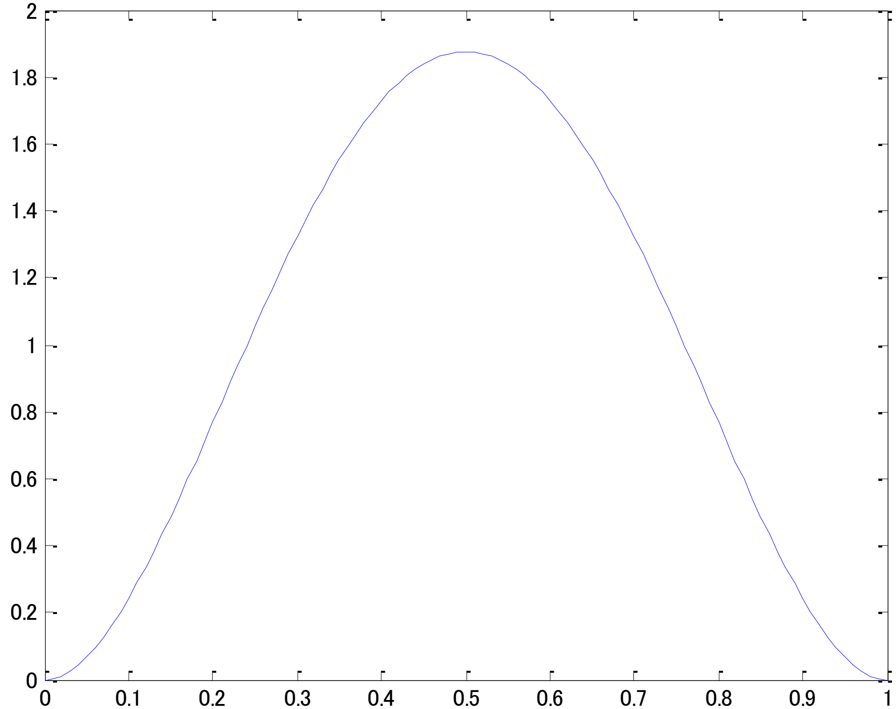
After normalizing it, we have

$$p(\theta|D_4, D_3, D_2, D_1) = 30(1-\theta)^2 \times \theta^2.$$

Posterior distribution

$$p(\theta|D_4, D_3, D_2, D_1) = 30(1-\theta)^{\,2} \times \theta^2$$



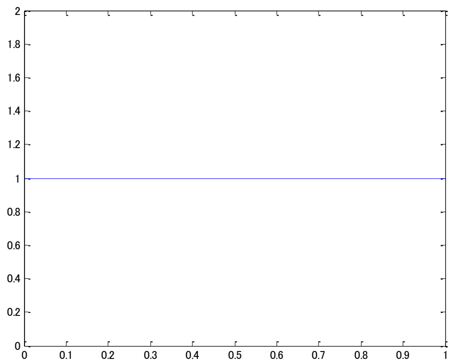


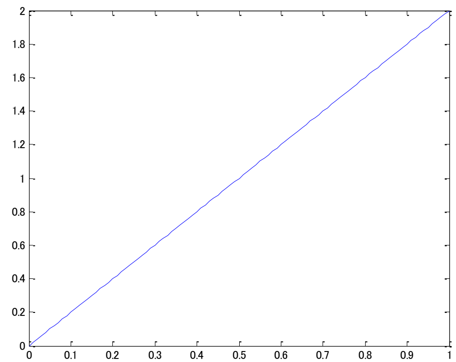
$$\text{Let } f(\theta) = 30(1-\theta)^{\,2} \times \theta^2 = 30(\theta^2 - 2\theta^3 + \theta^4)$$

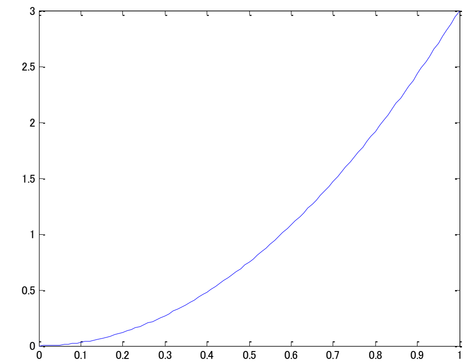
$$f'(\theta) = 30(2\theta - 6\theta^2 + 4\theta^3) = 60\theta(1 - 3\theta + 2\theta^2) = 120\theta\left(\theta - \frac{1}{2}\right)(\theta - 1).$$

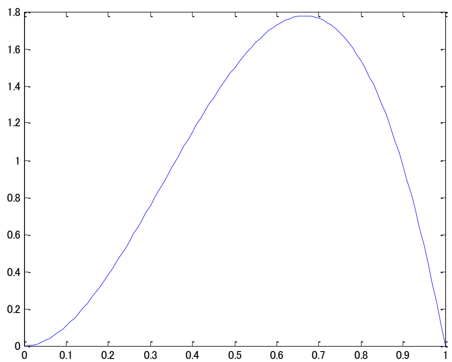
$$f'(\theta) = 0 \Leftrightarrow \theta = 0, \frac{1}{2}, 1.$$

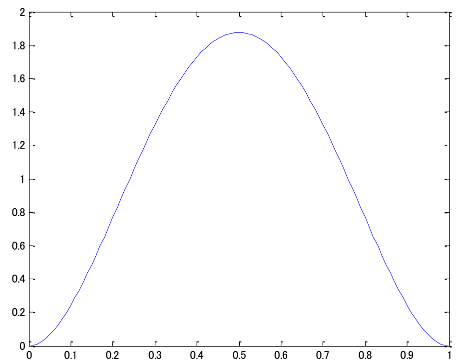
Data: H H T T











We can integrate old and new data by Bayes' theorem. It is difficult in the traditional statistics.